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SUMMARY

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The origin of "secular" behavior for the nonlinear cold electron plasma equations is studied. The equations involved are closely related to the Klein-Gordon equation with a small nonlinear term. A method is developed for arriving at perturbation theoretic solutions of this equation, and the method is then applied to the case of the higher order effects of an electromagnetic wave propagating in the cold electron plasma. An explicit expression for the second order frequency shift is calculated.

Author

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INTRODUCTION

When attempts are made to calculate the behavior of nonlinear mechanical systems in perturbation theory which goes beyond the linearized approximation, the difficulty of "secular" behavior often (though by no means always) appears. The higher orders contain, in addition to trigonometric terms, time-proportional or "secular" terms. The unbounded character of these terms soon invalidates the perturbation theoretic assumptions of smallness on which they were derived.

A systematic program for doing a type of perturbation theory that is free of secular terms in the case of the harmonic oscillator equation with a "small" nonlinear term was given some time ago by Krylov and Bogolyubov (Reference 1), and later refined and mathematically justified by Bogolyubov and Mitropolskii (Reference 2).** Recently, considerable interest has arisen in modifying these techniques to deal with partial differential and differentio-integral equations, especially in connection with the work of Frieman (Reference 3) and Sandri (Reference 4). Earlier calculations, which are more closely related to this work, were made by Jackson (Reference 5) and Sturrock (Reference 6).‡

The system treated here, a partial differential equation, is considerably simpler than those described in References 3 and 4. It is nontheless believed to be of value in that it illuminates, in a relatively uncluttered way, some new features which emerge as a consequence of its being a partial differential, rather than an ordinary differential, system.

The equation treated by Bogolyubov and Mitropolskii is

$$\left(\frac{d^2}{dt^2} + \omega^2\right) x(t) = \epsilon F\left(x, \frac{dx}{dt}\right) \tag{1}$$

*To be published in *Physics of Fluids*.
†Permanent address: University of Maryland, College Park, Maryland.
**A quite similar technique was put forward independently by M. V. Lighthill: "A Technique for Rendering Approximate Solutions to Physical Problems Uniformly Valid," *Phil. Mag.* Series 7, 40(311):1179-1201, December 1949.
‡See also Dolph, C. L., "A Unified Theory of the Nonlinear Oscillations of a Cold Plasma," *J. Math. Anal. Appl.* 5(1):94-118, August 1962.

where ϵ is a formal expansion parameter (eventually to be set equal to one) written to indicate the relative "smallness" of the right hand side; $F(x, dx/dt)$ is a known nonlinear functional of x and dx/dt; and ω^2 is a constant.

We shall examine the equation

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} + \lambda^2\right) f(x,t) = \epsilon F\left(f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}\right) \tag{2}$$

where ϵ is again the formal expansion parameter; F is again a known function (which depends on the problem under consideration); and c^2 and λ^2 are non-negative real constants.

The immediate motivation for studying Equation 2 was that the equations of motion for a "cold" electron plasma in a uniform positive background may be put in this form, component by component. But it also has the value of being sufficiently simple that explicit zeroth order ($\epsilon = 0$) solutions of it can be constructed which *both do and do not* lead to secular behavior when a straightforward attempt is made to extend them to higher order in ϵ. This simple example makes clear one difference between Equation 1 and an equation of the type 2. In general, for a given F, any $\epsilon = 0$ solution of Equation 1 will give rise to secularity when extended to higher order, whereas the conditions under which Equation 2 gives rise to secular behavior beyond the first order *depend sensitively upon which* $\epsilon = 0$ *solution it is that one wishes to extend to higher order in* ϵ*, even for a given* F.

We shall first give an example of both secular and non-secular behavior for Equation 2, and show how the difficulties may be remedied in the secular case. Then we shall give a simple physical example, that of the second order behavior of a nonlinear electromagnetic wave in a "cold" electron plasma.

THE KLEIN-GORDON EQUATION WITH A SMALL NONLINEAR TERM

A Single Monochromatic Wave

We shall first consider perturbations about the following $\epsilon = 0$ solution to Equation 2:

$$f = a \cos\left(K_0 x - \omega_0 t + \phi\right) \tag{3}$$

where

$$\omega_0^2 = c^2 K_0^2 + \lambda^2 \tag{4}$$

and, for $\epsilon = 0$, the quantities K_0, ω_0, a, and ϕ are constants.

This happens to be a situation for which secularity arises, so we shall set up the necessary formalism for handling it from the beginning. Following Reference 2, we seek a solution to

Equation 2 of the form

$$f = a\cos\psi + \epsilon\, u_1(a,\psi) + \epsilon^2 u_2(a,\psi) + \cdots , \tag{5}$$

where the amplitude a is now determined as a "slowly-varying" function of x and t by the relations

$$\frac{\partial a}{\partial t} = \epsilon A_1(a) + \epsilon^2 A_2(a) + \cdots , \tag{6a}$$

$$\frac{\partial a}{\partial x} = \epsilon D_1(a) + \epsilon^2 D_2(a) + \cdots . \tag{6b}$$

Here, ψ is a new "phase" variable, to be chosen to coincide with the phase of Equation 3 for $\epsilon = 0$:

$$\frac{\partial\psi}{\partial t} = -\omega_0 + \epsilon B_1(a) + \epsilon^2 B_2(a) + \cdots , \tag{7a}$$

$$\frac{\partial\psi}{\partial x} = K_0 + \epsilon C_1(a) + \epsilon^2 C_2(a) + \cdots . \tag{7b}$$

The (as yet undetermined) functions $A_1, A_2, \cdots, B_1, B_2, \cdots, C_1, C_2, \cdots, D_1, D_2, \cdots$, are to be *chosen* so as to render the solution (5) free from secular terms. Here, "secular" must be interpreted to mean ψ-proportional; i.e., solutions can break down because of linear growth in x as well as in t. The functions $u_1, u_2, \cdots$, are to be *periodic* in ψ.

The program is to express the various terms in Equation 2, by means of the relations (5), (6), and (7), in terms of the u's, A's, B's, C's, and D's, as functions of a and ψ. For brevity's sake, we shall first only go to $0(\epsilon)$, though in principle the method may be carried to any order in ϵ. Generally, the algebra becomes prohibitive beyond $0(\epsilon^2)$. Hereafter, the notation "$+ \cdots$" will mean "of higher order in ϵ".

The result of differentiating f with respect to x and t and using (6) and (7) is:

$$\begin{aligned}\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} + \lambda^2 f = {} & a\cos\psi\left(-\omega_0^2 + c^2 K_0^2 + \lambda^2\right) \\ & + \epsilon\left\{\lambda^2\left(\frac{\partial^2 u_1}{\partial\psi^2} + u_1\right) + 2\left(\omega_0 A_1 + c^2 K_0 D_1\right)\sin\psi \right. \\ & \left. + 2a\left(\omega_0 B_1 + c^2 K_0 C_1\right)\cos\psi\right\} \\ & + 0\left(\epsilon^2\right) .\end{aligned} \tag{8}$$

The zeroth order part of (8) vanishes identically, by virtue of (4). The coefficient of ϵ is to be equated to $F(f, \partial f/\partial t, \partial f/\partial x)$, with the arguments replaced by their zeroth order values:

$$f \to a\cos\psi ,$$

$$\partial f/\partial t \to \omega_0 a \sin\psi ,$$

$$\partial f/\partial x \to -K_0 a \sin\psi .$$

For the purpose of finding u_1, it is most convenient to write F as a Fourier series in ψ:

$$F\left(a\cos\psi,\ \omega_0 a\sin\psi,\ -K_0 a\sin\psi\right) = g_0(a) + \sum_{n=1}^{\infty}\left(g_n(a)\cos n\psi + f_n(a)\sin n\psi\right) . \tag{9}$$

The $f_n(a)$ and $g_n(a)$ are *known* functions of the amplitude a, determined by the functional form of F. We shall make the assumption—always satisfied in practice—that the $f_n(a)$ and $g_n(a)$ go to zero, as a goes to zero, at least as fast as $0\left(a^2\right)$.

The equation for u_1 may be written as:

$$\begin{aligned}\lambda^2\left(\frac{\partial^2}{\partial\psi^2} + 1\right)u_1 = g_0(a) &+ \left[f_1(a) - 2\omega_0 A_1 - 2c^2K_0D_1\right]\sin\psi \\ &+ \left[g_1(a) - 2a\omega_0 B_1 - 2ac^2K_0C_1\right]\cos\psi \\ &+ \sum_{n=2}^{\infty}\left(g_n(a)\cos n\psi + f_n(a)\sin n\psi\right) .\end{aligned} \tag{10}$$

This is, effectively, an ordinary differential equation in ψ of a standard type. Its solution contains terms proportional to $\psi\sin\psi$ and $\psi\cos\psi$ *unless* the coefficients of $\sin\psi$ and $\cos\psi$ on the right hand side of (10) vanish. *If* they vanish, then the general solution may be found in the form

$$u_1 = v_0(a) + \sum_{n=1}^{\infty}\left(v_n(a)\cos n\psi + w_n(a)\sin n\psi\right) . \tag{11}$$

Our objective is to find a solution free from ψ-proportional terms. This impels us to choose A_1, B_1, C_1, D_1, so that

$$2\left(\omega_0 A_1 + c^2K_0D_1\right) = f_1(a) , \tag{12a}$$

$$2a\left(\omega_0 B_1 + c^2K_0C_1\right) = g_1(a) . \tag{12b}$$

Two more relations may be deduced from the conditions

$$\frac{\partial^2\psi}{\partial t\,\partial x} = \frac{\partial^2\psi}{\partial x\,\partial t}$$

and

$$\frac{\partial^2 a}{\partial t\,\partial x} = \frac{\partial^2 a}{\partial x\,\partial t} .$$

Reference to (6) and (7) shows that, to lowest significant order, these become

$$A_1 \frac{dC_1}{da} = D_1 \frac{dB_1}{da}, \tag{13a}$$

$$A_1 \frac{dD_1}{da} = D_1 \frac{dA_1}{da}. \tag{13b}$$

Equations 12 and 13 are four relations for the four unknowns A_1, B_1, C_1, D_1, and once they have been obtained, (10) can be solved in a straightforward way by means of (11), and u_1 will contain no ψ-proportional terms. The function B_1 can be interpreted as a "frequency shift" and C_1 as a "wave number shift."

However, it is impossible to solve (12) and (13) completely without specifying the physical problem in more detail. For instance, if we wish to work a *boundary value* problem in which f is required to oscillate sinusoidally at a given x for all t, we do not expect a or ψ to vary with t, and B_1 and A_1 may be set equal to zero. This leaves

$$C_1 = \frac{g_1(a)}{2c^2 K_0 a} \tag{14a}$$

as the "wave number shift" and

$$D_1 = \frac{f_1(a)}{2c^2 K_0}. \tag{14b}$$

If we are interested in an *initial value* problem in which a pure sine wave is given for all x at $t = 0$, we may correspondingly set C_1 and D_1 equal to zero and solve for A_1 and B_1. Other choices are necessitated by still other problems. (Note in passing that in both cases, the corrections approach zero as a approaches zero, as they must in order to make physical sense.)

All this has been for a completely general F, imagined to contain all harmonics in ψ. In practice, it often happens that F contains only a few harmonics. Observe that secularity may *not* arise for some forms of the nonlinear term. Thus, if F is proportional to $(f)^2$ or $f\partial f/\partial x$, say, then $f_1(a)$ and $g_1(a)$ will both vanish; and in this order, we may set A_1, B_1, C_1, D_1, all identically zero, which is equivalent to a completely straightforward kind of perturbation theory. On the other hand, if we were to have F proportional to $(f)^3$, $f_1(a)$ is still zero, but $g_1(a) = 3a^3/4$ and a straightforward perturbation theory no longer works. The occurrence of secularity thus depends, in a given order, on the particular form of the nonlinear term. In the following sub-section, we shall show that, for a given F secularity likewise may or may not appear, depending on the choice of the $\epsilon = 0$ solution.

To close this discussion, we give the full solution for $u_1(a,\psi)$ of Equation 11. If we make the (arbitrary) choice that all the first harmonic shall be collected in lowest order, we may choose

$v_1(a) = w_1(a) = 0.$ Then for $n \neq 1$,

$$\left.\begin{aligned} v_n(a) &= \frac{g_n(a)}{\lambda^2\left(-n^2+1\right)} , \\ w_n(a) &= \frac{f_n(a)}{\lambda^2\left(-n^2+1\right)} , \end{aligned}\right\} \tag{15a}$$

$$\begin{pmatrix} f_n(a) \\ g_n(a) \end{pmatrix} = \frac{1}{\pi}\int_0^{2\pi} \begin{pmatrix} \sin n\psi \\ \cos n\psi \end{pmatrix} F\left(a\cos\psi,\ \omega_0 a\sin\psi,\ -K_0 a\sin\psi\right) d\psi \ . \tag{15b}$$

When the phenomenon of secularity occurs, one must go to one higher order in ϵ to get corrections to f which are uniformly valid through terms of $0(\epsilon)$ for changes in x and t of $0(1/\epsilon)$, and so the solution (15) is not of great interest by itself (Reference 2). However, the expressions such as (14) for frequency and wave number shifts *are* accurate to $0(\epsilon)$, and are usually more accessible to measurement than the expression for u_1, in any case.

Two Coupled Monochromatic Waves of Differing Frequencies

We now proceed to the case in which Equation 2 has the $\epsilon = 0$ solution

$$f = a\left[\cos\left(K_1 x - \omega_1 t + \phi_1\right) + \cos\left(K_2 x - \omega_2 t + \phi_2\right)\right] , \tag{16}$$

where

$$\omega_1^2 - c^2K_1^2 - \lambda^2 = \omega_2^2 - c^2K_2^2 - \lambda^2 = 0 , \tag{17}$$

but where there is not necessarily any other relationship between ω_1, K_1 and ω_2, K_2. If we were to do a straightforward perturbation theory, assuming

$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots , \tag{18}$$

with $f^{(0)}$ given by (16), it is clear that we can always find a secularity-free perturbation theoretic solution to (2) of the form

$$f^{(n)} = \sum_{r,s=-\infty}^{\infty} f_{rs}^{(n)}(a) \exp i\left[r\left(K_1 x - \omega_1 t + \phi_1\right) + s\left(K_2 x - \omega_2 t + \phi_2\right)\right] \tag{19}$$

except when there happen to exist two integers $\tilde{r}$, $\tilde{s}$ such that

$$\lambda^2\left(-\tilde{r}^2 - \tilde{s}^2 + 1\right) - 2\tilde{r}\tilde{s}\left(\omega_1\omega_2 - c^2K_1K_2\right) = 0 . \tag{20}$$

When (20) is fulfilled $f^{(1)}$, like the u_1 of Equation 10, will in general contain terms proportional to the phases, as well as trigonometric terms.

For a randomly selected pair of values ω_1, K_1 and ω_2, K_2, there will exist no integers $\tilde{r}$, $\tilde{s}$ which fulfill (20), and there is no necessity for the Bogolyubov techniques. When, on the other hand, two such integers $\tilde{r}$, $\tilde{s}$, do exist [the relation (20) will also hold for $-\tilde{r}$, $-\tilde{s}$, of course], then the Bogolyubov techniques are called for. One method of introducing them is the following, though it is not the only method.* Seek a solution of the form

$$f = a\left(\cos\psi_1 + \cos\psi_2\right) + \epsilon u_1\left(a, \psi_1, \psi_2\right) + \cdots$$

where now the amplitude a and the phases ψ_1 and ψ_2 develop according to:

$$\left.\begin{aligned}
\frac{\partial a}{\partial t} &= \epsilon A\left(a, \psi_1, \psi_2\right) + \cdots, \\
\frac{\partial a}{\partial x} &= \epsilon B\left(a, \psi_1, \psi_2\right) + \cdots, \\
\frac{\partial \psi_{1,2}}{\partial t} &= -\omega_{1,2} + \epsilon C_{1,2}\left(a, \psi_1, \psi_2\right) + \cdots, \\
\frac{\partial \psi_{1,2}}{\partial x} &= K_{1,2} + \epsilon D_{1,2}\left(a, \psi_1, \psi_2\right) + \cdots.
\end{aligned}\right\} \tag{21}$$

The left hand side of (2) may again be computed, by using the relations (21), to give

$$\begin{aligned}
\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} + \lambda^2 f = {} & a\left[\left(-\omega_1^2 + c^2K_1^2 + \lambda^2\right)\cos\psi_1 + \left(-\omega_2^2 + c^2K_2^2 + \lambda^2\right)\cos\psi_2\right] \\
& + \epsilon\left\{\sin\psi_1\left[2\omega_1 A + a\omega_1\frac{\partial C_1}{\partial\psi_1} + a\omega_2\frac{\partial C_1}{\partial\psi_2}\right.\right. \\
& \left. + 2c^2K_1B + ac^2K_1\frac{\partial D_1}{\partial\psi_1} + ac^2K_2\frac{\partial D_1}{\partial\psi_2}\right] + \sin\psi_2\left[2\omega_2 A\right. \\
& \left. + a\omega_1\frac{\partial C_2}{\partial\psi_1} + a\omega_2\frac{\partial C_2}{\partial\psi_2} + 2c^2K_2B + ac^2K_1\frac{\partial D_2}{\partial\psi_1} + ac^2K_2\frac{\partial D_2}{\partial\psi_2}\right] \\
& + \cos\psi_1\left[2a\omega_1C_1 - \omega_1\frac{\partial A}{\partial\psi_1} - \omega_2\frac{\partial A}{\partial\psi_2} + 2ac^2K_1D_1 - c^2K_1\frac{\partial B}{\partial\psi_1}\right. \\
& \left. - c^2K_2\frac{\partial B}{\partial\psi_2}\right] + \cos\psi_2\left[2a\omega_2C_2 - \omega_1\frac{\partial A}{\partial\psi_1} - \omega_2\frac{\partial A}{\partial\psi_2} + 2c^2aK_2D_2\right. \\
& \left.\left. - c^2K_1\frac{\partial B}{\partial\psi_1} - c^2K_2\frac{\partial B}{\partial\psi_2}\right]\right\} + \epsilon\left\{\lambda^2\left[\frac{\partial^2 u_1}{\partial\psi_1^2} + \frac{\partial^2 u_1}{\partial\psi_2^2} + u_1\right]\right. \\
& \left. + 2\left[\omega_1\omega_2 - c^2K_1K_2\right]\frac{\partial^2 u_1}{\partial\psi_1\partial\psi_2}\right\} + 0\left(\epsilon^2\right) .
\end{aligned} \tag{22}$$

*For simplicity, we assume only one independent pair of such integers, $\tilde{r}$, $\tilde{s}$, exists.

The zeroth order part of (22) vanishes identically. The first order part now must be equated to $\epsilon F\left[a\left(\cos\psi_1 + \cos\psi_2\right)\ ,\ a\left(\omega_1 \sin\psi_1 + \omega_2 \sin\psi_2\right),\ -a\left(K_1 \sin\psi_1 + K_2 \sin\psi_2\right)\right]$. It is most convenient to write this as a complex Fourier series,

$$F = \sum_{m,n=-\infty}^{+\infty} F_{mn}(a)\, e^{i\left(m\psi_1 + n\psi_2\right)} \tag{23}$$

where $F_{mn}{}^{*} = F_{-m,-n}$, since we deal only with real quantities. Calling everything in the first set of braces in Equation 22 "$G\left(a, A, B, C_{1,2}, D_{1,2}\right)$," we may finally write the equation for u_1 as

$$\lambda^2\left(\frac{\partial^2 u_1}{\partial\psi_1^2} + \frac{\partial^2 u_1}{\partial\psi_2^2} + u_1\right) + 2\left(\omega_1\omega_2 - c^2 K_1 K_2\right)\frac{\partial^2 u_1}{\partial\psi_1 \partial\psi_2}$$

$$= -\,G\left(a, A, B, C_{1,2}, D_{1,2}\right) + \sum_{m,n=-\infty}^{+\infty} F_{mn}(a) e^{i\left(m\psi_1 + n\psi_2\right)}. \tag{24}$$

This equation, analogously to (10), has a readily obtainable secularity-free solution of the form

$$u_1\left(\psi_1, \psi_2, a\right) = \sum_{m,n=-\infty}^{+\infty} u_{mn}(a) e^{i\left(m\psi_1 + n\psi_2\right)} \tag{25}$$

if and only if the functions $A, B, C_{1,2}, D_{1,2}$ *are chosen to depend on* ψ_1 *and* ψ_2 *in such a manner that:*

$$\oint d\psi_1 \oint d\psi_2\ e^{-i\left(\bar{r}\psi_1 + \bar{s}\psi_2\right)} G\left(a, A, B, C_{1,2}, D_{1,2}\right) - F_{\bar{r}\,\bar{s}}(a) = 0\ . \tag{26}$$

Since (26) involves complex numbers, it really amounts to two real equations upon $A, B, C_{1,2}, D_{1,2}$, which are linear partial differential equations with periodic coefficients.† We need not write them down in full detail, since they are most cumbersome, and not of interest for our purposes here.

From the requirements that

$$\partial^2 a/\partial x\, \partial t = \partial^2 a/\partial t\, \partial x\ ,\quad \partial^2\psi_{1,2}/\partial x\, \partial t = \partial^2\psi_{1,2}/\partial t\, \partial x\ ,$$

three additional conditions may be adduced. They are:

$$K_1 \frac{\partial A}{\partial\psi_1} + K_2 \frac{\partial A}{\partial\psi_2} = -\left(\omega_1 \frac{\partial B}{\partial\psi_1} + \omega_2 \frac{\partial B}{\partial\psi_2}\right), \tag{27a}$$

$$K_1 \frac{\partial D_1}{\partial\psi_1} + K_2 \frac{\partial D_1}{\partial\psi_2} = -\left(\omega_1 \frac{\partial C_1}{\partial\psi_1} + \omega_2 \frac{\partial C_1}{\partial\psi_2}\right), \tag{27b}$$

$$K_1 \frac{\partial D_2}{\partial\psi_1} + K_2 \frac{\partial D_2}{\partial\psi_2} = -\left(\omega_1 \frac{\partial C_2}{\partial\psi_1} + \omega_2 \frac{\partial C_2}{\partial\psi_2}\right) \tag{27c}$$

†We may consider A, B, · · · · as being expressed as Fourier series in ψ_1, ψ_2, in which case (26) leads to algebraic conditions on the Fourier coefficients.

Satisfaction of these five relations on the six quantities A, B, $C_{1,2}$, $D_{1,2}$, then, will guarantee that the solution shall be secularity-free. It is clear that considerable latitude is left to choose the functions conveniently for whatever problem is under consideration. The point to be emphasized, however, is that secularity is by no means inherent in the equation to be studied, or in the form of the interaction term. Nor does its origin necessarily have a simple physical interpretation as it does for Equation 1, where the presence of secular terms is simply interpretable as a *resonance* between one of the frequencies present in the nonlinear coupling term and one of the natural frequencies of the system. It would take considerably more insight to apprehend the physical meaning of Equation 20, or any connection it might have with the secularity condition for Equation 10.

THE COLD ELECTRON PLASMA

Suppose we consider a cold (no thermal motions) electron plasma of equilibrium number density n_0, moving in a uniform positive background, assumed immobile. If we make a perturbation about a uniform, field-free equilibrium, the appropriate variables for describing the system are:

$\vec{v}$ = electron velocity,

$-e\left(n_0 + n(\vec{x}, t)\right)$ = electron charge density,

$+en_0$ = positive background charge density,

$\vec{E}, \vec{B}$ = electric and magnetic fields.

All these will be treated as perturbations—i.e., as first order in the amplitude—except for n_0.

The dynamical equations are well known:

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial \vec{x}} \cdot (n\vec{v}) + \frac{\partial}{\partial \vec{x}} \cdot \left(n_0 \vec{v}\right) = 0 , \tag{28a}$$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \frac{\partial}{\partial \vec{x}} \vec{v} = -\frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{v} \times B\right) , \tag{28b}$$

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial \vec{x}} \times \vec{E} , \tag{28c}$$

$$\frac{1}{c}\frac{\partial \vec{E}}{\partial t} = \frac{\partial}{\partial \vec{x}} \times \vec{B} + \frac{4\pi e}{c}\left(n_0 + n\right)\vec{v} , \tag{28d}$$

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{E} = -4\pi e n , \tag{29a}$$

$$\frac{\partial}{\partial \vec{x}} \cdot \vec{B} = 0 . \tag{29b}$$

Equations 29 can be regarded as initial conditions; once fulfilled, they are preserved by Equations 28.

A small amount of algebraic juggling shows that Equations 28 lead to

$$\frac{\partial^2}{\partial t^2}\vec{E} + c^2 \frac{\partial}{\partial \vec{x}} \times \left(\frac{\partial}{\partial \vec{x}} \times \vec{E}\right) + \omega_{pe}^2 \vec{E}$$

$$= -\epsilon\, 4\pi e\left[n_0\left(\vec{v}\cdot\frac{\partial}{\partial \vec{x}}\vec{v} + \frac{e}{mc}\vec{v}\times\vec{B}\right) - \frac{\partial}{\partial t}(n\vec{v})\right], \tag{30a}$$

$$\frac{\partial^2}{\partial t^2}\frac{\partial \vec{B}}{\partial t} + c^2 \frac{\partial}{\partial \vec{x}} \times \left(\frac{\partial}{\partial \vec{x}} \times \frac{\partial \vec{B}}{\partial t}\right) + \omega_{pe}^2 \frac{\partial \vec{B}}{\partial t}$$

$$= \epsilon\left[4\pi e n_0 c \frac{\partial}{\partial \vec{x}} \times \left(\vec{v}\cdot\frac{\partial}{\partial \vec{x}}\vec{v} + \frac{e\vec{v}}{mc}\times\vec{B}\right) - 4\pi e c \frac{\partial}{\partial \vec{x}} \times \frac{\partial}{\partial t}(n\vec{v})\right]. \tag{30b}$$

The formal expansion parameter ϵ has been written on the right hand side of Equations 30 only to remind us that these terms are "small" in the sense of being second order in the amplitude. The quantity ω_{pe}^2 is $4\pi n_0 e^2/m$, the plasma frequency.

If we now restrict ourselves to disturbances which are functions of only one spatial dimension (x, say), we may write the expression (30a) in the form

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} + \omega_{pe}^2 + c^2\delta_{1,i}\frac{\partial^2}{\partial x^2}\right)E_i = \epsilon\mathcal{F}_i\ , \tag{31}$$

where $\delta_{1,i} = 0$ unless $i = 1$, and $\delta_{1,1} = 1$; $\mathcal{F}_i$ represents the i^{th} component of the right hand side of (30a). Since (30b) is just the curl of (30a), it is not necessary to write it in the form of (31). It matters not at all that we have not bothered to express the right hand side of (30a) as a function of $\vec{E}$ alone, since we always need the values of the quantities which enter into $\mathcal{F}_i$ to one lower order than those under consideration on the left hand side of (30a); and those will be shown to be obtainable in each order directly from (28).

The $\epsilon = 0$ solution to (31) is just the standard cold-plasma set of field-free normal modes, which is well understood (Reference 7). It will be apparent that each component of (31) has the form of Equation 2 and is therefore immediately susceptible to the methods already discussed. Equation 31 represents a generalization of the equation treated by Jackson (Reference 5), which, however, does not exhibit all the features of (31) because of the absence of x-derivatives on the left hand side in the purely electrostatic case, and of the fact that the x and t dependences separate.

The number of possibilities from (31) is very large, due to the wide range of choices for the $\epsilon = 0$ solution. The simplest case—beyond that of a pure electrostatic oscillation (Reference 5)—is that of a pure transverse, linearly polarized, electromagnetic wave. We shall extend this solution to the next two orders above the linear approximation in the following paragraphs.

Calling our base vectors $\hat{e}_x$, $\hat{e}_y$, $\hat{e}_z$, the appropriate zeroth order solution is

$$\vec{E} = E_0 \hat{e}_y \cos\left(k_0 x - \omega_0 t\right) ,$$

$$\vec{B} = \frac{ck_0}{\omega_0} E_0 \hat{e}_z \cos\left(k_0 x - \omega_0 t\right)$$

$$\vec{v} = \frac{eE_0}{m\omega_0} \hat{e}_y \sin\left(k_0 x - \omega_0 t\right)$$

$$n = 0 \tag{32}$$

where

$$\omega_0^2 = \omega_{pe}^2 + c^2 k_0^2 \tag{33}$$

This zeroth order solution for $\vec{E}$ is conveniently written:

$$E_i = a\left[\phi_i\, e^{i\left(k_0 x - \omega_0 t\right)} + \phi_i^*\, e^{-i\left(k_0 x - \omega_0 t\right)}\right]$$

where ϕ is a column vector

$$\phi = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix} \tag{34}$$

and $a = E_0$ is the amplitude.

The results of Section II suggest that we seek a solution of the form

$$E_i = a\left(\phi_i\, e^{i\psi} + \phi_i^*\, e^{-i\psi}\right) + \epsilon u_i^{(1)}(a,\psi) + \epsilon^2 u_i^{(2)}(a,\psi) + \cdots \tag{35}$$

where now the amplitude a and phase ψ are to vary in x and t according to

$$\left.\begin{aligned}
\frac{\partial a}{\partial t} &= \epsilon A_1(a) + \epsilon^2 A_2(a) + \cdots , \\
\frac{\partial a}{\partial x} &= \epsilon D_1(a) + \epsilon^2 D_2(a) + \cdots , \\
\frac{\partial \psi}{\partial t} &= -\omega_0 + \epsilon B_1(a) + \epsilon^2 B_2(a) + \cdots , \\
\frac{\partial \psi}{\partial x} &= k_0 + \epsilon C_1(a) + \epsilon^2 C_2(a) + \cdots ,
\end{aligned}\right\} \tag{36}$$

with the functions A_1, B_1, $\cdots$, as yet undetermined.

The left hand side of (31), expressed in terms of these functions, becomes, for $i = 1$,

$$\left(\frac{\partial^2}{\partial t^2} + \omega_{pe}^2\right) E_1 = \epsilon\left\{\omega_0^2 \frac{\partial^2 u_1^{(1)}}{\partial\psi^2} + \omega_{pe}^2 u_1^{(1)} + 2\left(-i\omega_0 A_1 + a\omega_0 B_1\right)\phi_1 e^{i\psi} + 2\left(i\omega_0 A_1 + a\omega_0 B_1\right) \phi_1^* e^{-i\psi}\right\} + 0\left(\epsilon^2\right) \quad ; \qquad (37)$$

and for $i = 2$ or 3,

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2} + \omega_{pe}^2\right) E_i = \epsilon\left\{\omega_{pe}^2\left(\frac{\partial^2 u_i^{(1)}}{\partial\psi^2} + u_i^{(1)}\right) + 2\left(-i\omega_0 A_1 + a\omega_0 B_1 - ic^2 k_0 D_1 + ac^2 C_1 k_0\right)\phi_i e^{i\psi} + 2\left(i\omega_0 A_1 + a\omega_0 B_1 + ic^2 k_0 D_1 + ac^2 C_1 k_0\right)\phi_i^* e^{-i\psi}\right\} + 0\left(\epsilon^2\right) . \qquad (38)$$

These three expressions are to be equated to the three components of the right hand sides of (31), with the quantities $\vec{v}$, $\vec{E}$, $\vec{B}$, n replaced by their values (32), and with the substitution $E_0 \to a$. The result can be conveniently written as a Fourier series

$$\mathcal{F}_i = \sum_{m=-\infty}^{+\infty} \Phi_i(m,a)\, e^{im\psi} , \qquad (39)$$

where the $\Phi_i(m,a)$ are *known* functionals of a.

Suppose we now consider whether secular terms can arise. We seek the solution for $u_i^{(1)}$ in the form

$$u_i^{(1)} = \sum_{-\infty}^{\infty} u_i^{(1)}(m,a)\, \epsilon^{im\psi} \qquad (40)$$

The equation for $u_i^{(1)}$ then becomes

$$\sum_{m=-\infty}^{\infty} \left[\left(-m^2+1\right)\omega_{pe}^2 + m^2\delta_{1,i}\left(\omega_{pe}^2 - \omega_0^2\right)\right] u_i^{(1)}(m,a)\, e^{im\psi}$$
$$= -2\left(-i\omega_0 A_1 + a\omega_0 B_1 - ic^2 k_0 D_1 + ac^2 C_1 k_0\right)\phi_i e^{i\psi} - 2\left(i\omega_0 A_1 + a\omega_0 B_1 + ic^2 k_0 D_1 + ac^2 C_1 k_0\right)\phi_i^* e^{-i\psi} + \sum_{m=-\infty}^{\infty} \Phi_i(m,a)\, e^{im\psi} , \qquad (41)$$

(we have made use of $\phi_1 = 0$).

We may always solve (41) for $u_i^{(1)}(m,a)$ with the absence of ψ-proportional terms, except when a coefficient of $u_i^{(1)}(m,a)$ on the left hand side of (41) vanishes for some m. Since $\omega_0^2 > \omega_{pe}^2$ it is clear that this can only occur when $m = \pm 1$, and only in the components $i = 2$ or 3. However, a term-by-term inspection of $\Phi_i(m,a)$ reveals that for $i = 2$ or 3, the quantities $\Phi_i(m,a)$ vanish identically for the choice (33) of the zeroth order values. Thus a result emerges which would have been hard to guess from the original set of equations (28). Secular behavior cannot arise in second order for this case, and we may set $A_1 = B_1 = C_1 = D_1 = 0$, and solve in a completely straightforward way for $u_i^{(1)}(m,a)$:

$$u_i^{(1)}(m,a) = \frac{\Phi_i(m,a)}{\left(-m^2+1\right)\omega_{pe}^2 + m^2\delta_{1,i}\left(\omega_{pe}^2-\omega_0^2\right)}\,, \quad m \neq \pm 1\,,$$

$$u_i^{(1)}(\pm 1,a) = 0\,. \tag{42}$$

The calculation of the coefficients of the Fourier coefficients is a matter of simple algebra, and we may write down in full the first correction to the linear solution (32):

$$\vec{u}^{(1)} \equiv E^{(1)} = E_0^2\,\frac{ek_0\omega_{pe}^2}{2m\omega_0^2}\,\frac{\sin 2\left(k_0x-\omega_0t\right)}{\left(3\omega_{pe}^2+4c^2k_0^2\right)}\,\hat{e}_x\,,$$

$$\vec{B}^{(1)} = 0\,,$$

$$n^{(1)} = -E_0^2\,\frac{e^2n_0k_0^2}{m^2\omega_0^2}\,\frac{\cos 2\left(k_0x-\omega_0t\right)}{\left(3\omega_{pe}^2+4c^2k_0^2\right)}\,,$$

$$\vec{v}^{(1)} = -E_0^2\,\frac{e^2k_0}{m^2\omega_0}\,\frac{\cos 2\left(k_0x-\omega_0t\right)}{\left(3\omega_{pe}^2+4c^2k_0^2\right)}\,\hat{e}_x \tag{43}$$

The only qualitatively new feature which shows up in $O(\epsilon)$, then, is a *longitudinal* electric field. It is only in $O(\epsilon^2)$ that secularity manifests itself and it becomes necessary to use the Bogolyubov methods. Making use of the $O(\epsilon)$ solution we have just computed, we may write down the $O(\epsilon^2)$ part of the left hand side of (31) in terms of the A's, B's, etc. We shall need:

$$\left\{\frac{\partial^2E_i}{\partial t^2} - c^2\frac{\partial^2E_i}{\partial x^2} + \omega_{pe}^2E_i\right\}_{O(\epsilon^2)\text{part}} = \epsilon^2\Bigg[\left(\omega_0^2-c^2k_0^2\right)\frac{\partial^2u_i^{(2)}}{\partial\psi^2} + \omega_{pe}^2u_i^{(2)}$$

$$+\left(-2i\omega_0A_2 + 2a\omega_0B_2 - 2ik_0c^2D_2 + 2ac^2k_0C_2\right)\phi_i\,e^{i\psi}$$

$$+\left(2i\omega_0A_2 + 2a\omega_0B_2 + 2ik_0c^2D_2 + 2ac^2k_0C_2\right)\phi_i^*\,e^{-i\psi}\Bigg]\,. \tag{44}$$

It is also a simple matter to compute the $0(\epsilon^2)$ part of the right hand side of (31) from (43) and (32). The result expressed in terms of a and ψ, is

$$\mathcal{F}_i\Big]_{0(\epsilon^2)\text{part}} = \frac{\omega_{pe}^2 k_0^2 e^2 a^3 \delta_{i,2}}{4\omega_0^2 m^2 \left(3\omega_{pe}^2 + 4c^2k_0^2\right)} \left\{3\left(e^{3i\psi} + e^{-3i\psi}\right) - \left(e^{i\psi} + e^{-i\psi}\right)\right\}. \tag{45}$$

Since the only nonvanishing component of (45) is in the 2-direction (y-direction), the only nonvanishing component of $u_i^{(2)}$ will also be $u_2^{(2)}$. The equation to be solved in the $0(\epsilon^2)$ approximation is, therefore:

$$\begin{aligned}\left(\omega_0^2 - c^2k_0^2\right)\frac{\partial^2 u_2^{(2)}}{\partial\psi^2} + \omega_{pe}^2 u_2^{(2)} = \\ &-\left(-2i\omega_0 A_2 + 2a\omega_0 B_2 - 2ik_0c^2D_2 + 2ak_0c^2C_2\right)\phi_2 e^{i\psi} \\ &-\left(2i\omega_0 A_2 + 2a\omega_0 B_2 + 2ik_0c^2D_2 + 2ac^2k_0C_2\right)\phi_2^* e^{-i\psi} \\ &+\frac{\omega_{pe}^2 k_0^2 e^2 a^3}{4\omega_0^2 m^2\left(3\omega_{pe}^2 + 4c^2k_0^2\right)}\left[3\left(e^{3i\psi} + e^{-3i\psi}\right) - \left(e^{i\psi} + e^{-i\psi}\right)\right].\end{aligned} \tag{46}$$

We may find a solution to (46) of the form

$$u_2^{(2)}(a,\psi) = \sum_{m=-\infty}^{\infty} u_2^{(2)}(a,m)e^{im\psi}, \tag{47}$$

which is free of ψ-proportional terms, if and only if the coefficients of $e^{i\psi}$ and $e^{-i\psi}$ on the right hand side of (46) vanish. This gives us two conditions on A_2, B_2, C_2, D_2, which are algebraic relations. Two more, analogous to Equations 13, are given by the requirements that

$$\frac{\partial^2 a}{\partial t\partial x} = \frac{\partial^2 a}{\partial x\partial t}$$

and

$$\frac{\partial^2 \psi}{\partial t\partial x} = \frac{\partial^2 \psi}{\partial x\partial t};$$

they are:

$$D_2\frac{dA_2}{da} = A_2\frac{dD_2}{da}, \tag{48a}$$

$$D_2\frac{dB_2}{da} = A_2\frac{dC_2}{da}. \tag{48b}$$

Let us now specialize the problem to one in which the spatial periodicity and amplitude are given and we are to calculate the "frequency shift." This means setting $C_2 = D_2 = 0$. The condition that the coefficients of $e^{i\psi}$ and $e^{-i\psi}$ on the right hand side of (46) vanish reduces to the equation

$$\left(i\omega_0 A_2 - a\omega_0 B_2\right) = \frac{\omega_{pe}^2 k_0^2 e^2 a^3}{4m^2 \omega_0^2 \left(3\omega_{pe}^2 + 4c^2 k_0^2\right)} \tag{49}$$

and its complex conjugate relation. We may therefore find $A_2 = 0$, and the frequency shift $\Delta\omega$ becomes (substituting E_0 for the amplitude a):

$$\Delta\omega = -B_2 = \frac{e^2 k_0^2 \omega_{pe}^2}{4m^2\omega_0^3} \frac{E_0^2}{\left(3\omega_{pe}^2 + 4c^2k_0^2\right)} \tag{50}$$

It does not seem worthwhile to write down the $0(\epsilon^2)$ corrections to $\vec{E}$, $\vec{B}$, $\vec{v}$, and n, though it would be easy to do so. In any experiment one might imagine, Equation 50 would probably be the easiest quantity to measure.

It is possible to calculate the frequency shift of a *standing* electromagnetic wave of given periodicity as well. One assumes a linearly polarized standing wave solution for (31) in lowest order and writes

$$E_i = a_i \sin k_0 x \cos\psi + \epsilon u_i^{(1)}(a, \psi, x) + \epsilon^2 u_i^{(2)}(a, \psi, x) + \cdots$$

Then, only the t-dependence of the phase variable and amplitude are expanded:

$$\frac{da}{dt} = \epsilon A_1(a) + \epsilon^2 A_2(a) + \cdots$$

$$\frac{d\psi}{dt} = \omega_0 + \epsilon B_1(a) + \epsilon^2 B_2(a) + \cdots$$

In lowest significant order, the frequency shift turns out to be exactly one eighth of the result given in Equation (50).

DISCUSSION

We have given a technique for obtaining uniformly valid, perturbation theoretic solutions to the Klein-Gordon equation with a small nonlinear term. The method has been applied to calculate the second order frequency shift of an electromagnetic wave in a cold electron plasma. The smallness of the expression (50) for attainable parameters is an indication of just how good an approximation the linear theory is at these frequencies.

It is not to be inferred, however, that the method adapts itself readily to all partial differential equations. For instance, the reader can easily convince himself that it fails for a nonlinear sound wave described by the Euler equations. (It appears to fail in all situations for which the $\epsilon = 0$ equation is a wave equation, $(\partial^2/\partial t^2 - c^2\partial^2/\partial x^2)\, f = 0$.) The physical reason is that, due to a steepening of the exact nonlinear wave front—obtainable from the Riemann invariants—a vertical tangent develops after a time of 0(1/ the amplitude). This destroys any regularity properties which may have existed in the original wave profile. Since the exact solution does not remain "close" to the $\epsilon = 0$ solution in *any* sense, after a time of order $1/\epsilon$, it is not surprising that perturbation theory is of little use beyond this time.

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